Notes 4. POWER SERIES

4.1 Power Series

Recall a series of functions is of the form $\sum_{j=j_0}^{\infty} f_j(x)$ where each f_j is a function defined on a common subset E of $\mathbb{R}$. In practice there are two kinds of series of functions which are important: Power series and trigonometric series. We shall study power series in these notes and leave trigonometric series to MATH3060 Mathematical Analysis III.

By a *power series* we mean a series of the form $\sum_{j=0}^{\infty} a_j(x-x_0)^j$ where $a_j \in \mathbb{R}$ and x_0 is a fixed point in $\mathbb{R}$. For instance,

$$\sum_{j=0}^{\infty} \frac{x^j}{j!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots ,$$

$$\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} x^{2j} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots ,$$

and

$$\sum_{j=3}^{\infty} j(x-2)^j = 3(x-2)^3 + 4(x-2)^4 + 5(x-2)^5 + \cdots ;$$

are power series. In the second example, it is understood that all $a_j = 0$ for odd j, and in the last example, $a_0 = a_1 = a_2 = 0$. One should keep in mind that inserting zeros between two summands of a series affects neither the convergence nor the final sum of the series. On the other hand,

$$\sum_{n=1}^{\infty} \frac{1}{n^2+x^2}, \quad \sum_{j=0}^{\infty} \frac{1}{j!} \cos jx, \quad \text{and} \quad \sum_{k=2}^{\infty} \frac{k!(e^{kx}+x^k)}{\log k},$$

are not power series.

Given a series of functions, we would like to determine its pointwise convergence and uniform convergence. For pointwise convergence, the tests that are shown in 9.1 - 9.3 of Textbook are applicable. For uniform convergence, Weierstrass M-Test and Cauchy criterion are the common tools. However, for power series we have a very general and yet precise result. To formulate it one needs to introduce the notion of the radius of convergence of a power series.

Let $\rho := \overline{\lim_{n\to\infty}} |a_n|^{1/n} \in [0,\infty]$ where $\overline{\lim_{n\to\infty}} x_n$ denotes the limit superior of a sequence $\{x_n\}$. (We learnt limit superior in last semester. Please look up your notes to refresh your memory. See also the exercise.) Define the *radius of convergence* of $\sum_{j=0}^{\infty} a_j(x-x_0)^j$ to be

$$R = \begin{cases} 0, & \text{if } \rho = \infty, \\ 1/\rho, & \text{if } \rho \in (0,\infty), \\ \infty, & \text{if } \rho = 0. \end{cases}$$

The following theorem is the main result for power series. In fact, it also holds for complex variables.

Theorem 4.1. ***(Cauchy-Hadamard Theorem)***

(a) When $R \in (0,\infty)$, the power series $\sum_{j=0}^{\infty} a_j(x-x_0)^j$ converges absolutely at every $x \in (x_0-R, x_0+R)$ and diverges at every x satisfying $|\,x-x_0\,| > R$. Moreover, the convergence is uniform on every subinterval $[a,b] \subset (x_0-R, x_0+R)$.

(b) When $R=\infty$, the power series converges absolutely at every $x \in \mathbb{R}$ and converges uniformly on any finite interval.

(c) When $R=0$, the power series diverges at every $x \in \mathbb{R} \setminus \{x_0\}$.

Proof. (a) We show that for any $r < R$, the series is absolutely and uniformly convergent on $[x_0-r, x_0+r]$. To this end we fix a small $\delta > 0$ such that $(\rho+\delta)r < 1$. This is possible because $\rho r = r/R < 1$. Then, as $\overline{\lim}\sqrt[n]{|\,a_n\,|} = \rho$, there exists n_0 such that $\sqrt[n]{|\,a_n\,|} \le \rho+\delta$, $\forall\, n \ge n_0$. For $x \in [x_0-r, x_0+r]$, we have

$$\begin{aligned} \sqrt[n]{|\,a_n(x-x_0)^n\,|} &= \sqrt[n]{|\,a_n\,|}\,|\,x-x_0\,| \\ &\le \sqrt[n]{|\,a_n\,|}\,r \\ &\le (\rho+\delta)r < 1, \quad \forall\, n \ge n_0. \end{aligned}$$

Taking $\alpha = (\rho+\delta)r$, we have $|\,a_n(x-x_0)^n\,| \le \alpha^n$ and $\sum_{n=0}^{\infty} \alpha^n < +\infty$. By Weierstrass M-Test we conclude that $\sum_{j=0}^{\infty} a_j(x-x_0)^j$ converges absolutely and uniformly on $[x_0-r, x_0+r]$.

When x_1 satisfies $|x_1-x_0| > R$, assume it is $x_1-x_0 > R$, say. Fix an $\varepsilon_0 > 0$ such that $(\rho-\varepsilon_0)(x_1-x_0) > 1$. This is possible because $\rho(x_1-x_0) > \rho R = 1$. There exists n_1 and a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that $|a_{n_j}|^{1/n_j} \ge \rho-\varepsilon_0$, $\forall\, n_j \ge n_1$. Then

$$\sqrt[n_j]{|\,a_{n_j}(x_1-x_0)^{n_j}\,|} = |\,a_{n_j}\,|^{1/n_j}(x_1-x_0) \ge (\rho-\varepsilon_0)(x_1-x_0) > 1.$$

It shows that $\{\,a_n(x_1-x_0)^n\,\}$ does not tend to zero, so the power series diverges.

The proofs of (b) and (c) are essentially contained in the above proof. We leave them as an exercise.

$\square$

We point out another way to evaluate the radius of convergence is by the formula

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

provided the limit exists. You may prove it as an exercise.

We observe that termwise differentiation and integration of a given power series yield power series:

$$\sum_{j=0}^{\infty}(j+1)a_{j+1}x^j \quad \text{and} \quad \sum_{j=1}^{\infty}\frac{a_{j-1}}{j}x^j.$$

It is routine to check that these two series have the same radius of convergence as the original one. (Indeed, suppose that $\rho = \lim_{j\to\infty}|a_j|^{1/j}$ exists and belongs to $(0,\infty)$. For $\varepsilon > 0$, there exists some n_0 such that $\rho - \varepsilon/2 \le |a_j|^{1/j} \le \rho - \varepsilon/2$, for all $n \ge n_0$. It follows that $(\rho - \varepsilon/2)^{1+1/j} \le |a_{j+1}|^{1/j} \le (\rho + \varepsilon/2)^{1+1/j}$. As $\lim_{j\to\infty}(\rho+\varepsilon/2)^{1/j} = 1$, we can find some $n_1 \ge n_0$ such that $(\rho+\varepsilon/2)^{1+1/j} \le \rho+\varepsilon$ and $\rho - \varepsilon \le (\rho - \varepsilon/2)^{1+1/j}$ for all $n \ge n_1$. Therefore, for these n, $|a_{j+1}^{1/j} - \rho| < \varepsilon$. We have shown that $\lim_{j\to\infty}|a_{j+1}|^{1/j} = \rho$. The j-th term in the series obtained from differentiation is given by $(j+1)a_{j+1}$. We have $\lim_{j\to\infty}|(j+1)a_{j+1}|^{1/j} = \lim_{j\to\infty}|j+1|^{1/j}\lim_{j\to\infty}|a_{j+1}|^{1/j} = \lim_{j\to\infty}|a_{j+1}|^{1/j} = \rho$, so this derived series has the same radius of convergence as the original one. The other cases can be treated similarly.) Since the partial sums of a power series are polynomials, in particular they are continuous on $[x - 0 - r, x_0 + r]$ for $r < R$. By uniform convergence the power series is a continuous function on $x_0 - R, x_0 + R)$. In fact, using the "exchange" theorems in Notes 3, we arrive at a strong conclusion.

Theorem 4.2. *Every power series is a smooth function on $(x_0 - R, x_0 + R)$. Moreover, termwise differentiations and integrations commute with the summation.*

Cauchy-Hadamard theorem says nothing on the convergence of a power series at its "boundary points". Let us consider an example.

Example 4.1. We start with the "mother" geometric series

$$\sum_{j=0}^{\infty}(-1)^j x^j = 1 - x + x^2 - x^3 + x^4 - \cdots$$

it is clear that its radius of convergence is equal to 1. Integrating both sides from

0 to $x \in (-1, 1)$, we get the "grandmother"

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1}x^j}{j} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots .$$

Integrating once more yields the "great grandmother"

$$\sum_{j=2}^{\infty} \frac{(-1)^{j-2}x^j}{(j-1)j} = \frac{x^2}{2} - \frac{x^3}{2\times 3} + \frac{x^4}{3\times 4} - \frac{x^5}{4\times 5} + \cdots .$$

On the other hand, by differentiating the mother we get the "child"

$$\sum_{j=0}^{\infty} (-1)^{j+1}(j+1)x^j = -1 + 2x - 3x^2 + 4x^3 - \cdots .$$

According to Cauchy-Hadamard theorem these series are all convergent in $(-1, 1)$ and divergent in $(-\infty, -1) \cup (1, \infty)$. At the boundary points 1 and -1 they could be convergent or divergent. Indeed, both the "mother" and the "child" diverge at the boundary points, the "grandmother" converges at 1 but diverges at -1, and the "great grandmother" converges at both ends. In general, when one family member is convergent at a boundary point, its ancestor is also convergent at the same point. (Why?) But the converse is not necessarily true.

From this example you can see that the convergence of a power series at the boundary points is a delicate matter and must be discussed case by case. However, there is a general result, namely, Abel's limit theorem. We know that a power series $s(x) \equiv \sum_{j=0}^{\infty} a_j (x - x_0)^j$ is smooth and in particular a continuous function on $(-R, R)$ when R is positive. In case the numerical series $\sum_{j=0}^{\infty} a_j R^j$ is convergent, one naturally wonders whether $s(x)$ extends to be a continuous function on $(-R, R]$ by setting $s(R) = \sum_{j=0}^{\infty} a_j R^j$. Similar question holds at $x = -R$. The following theorem gives an affirmative answer to this question.

The rest of this section is for optional reading.

Theorem 4.3. ***(Abel's Limit Theorem)*** *Let the radius of convergence of $\sum_{j=0}^{\infty} a_j (x - x_0)^j$ be $R \in (0, \infty)$. If $\sum_{j=0}^{\infty} a_j R^j$ is convergent, then*

$$\lim_{x \to R^-} s(x) = \sum_{j=0}^{\infty} a_j R^j$$

Proof. Without loss of generality we take $x_0 = 0$ and $R = 1$. Let $s_n(x) = \sum_{j=0}^{n} a_j x^j$, $s(x) = \sum_{j=0}^{\infty} a_j x^j$, for $x \in (-R, R)$, $s_n = \sum_{j=0}^{n} a_j$ and $s^* = \sum_{j=0}^{\infty} a_j$.

(Because the summation starts from zero, the n-th partial sum starts at $s_0 = a_0$.) We have the identity

$$(1-x)\sum_{j=0}^{\infty} s_j x^j = \sum_{j=0}^{\infty} a_j x^j, \ x \in (-1,1)$$

which could be verified directly. Taking advantage of this identity we have

$$\begin{aligned} s(x) - s^* &= (1-x)\sum_{j=0}^{\infty} s_j x^j - (1-x)s^* \sum_{j=0}^{\infty} x^j \\ &= (1-x)\sum_{j=0}^{\infty} (s_j - s^*)x^j . \end{aligned}$$

As $s_n \to s^*$, for every $\varepsilon > 0$, there exists N such that $|s_n - s^*| < \varepsilon/2$ for any $n \geq N$. Therefore, for $x \in (0,1)$,

$$\begin{aligned} |s(x) - s^*| &\leq (1-x)\sum_{j=0}^{N} |s_j - s^*| x^j + (1-x)\left|\sum_{j=N+1}^{\infty} (s_j - s^*)x^j\right| \\ &\leq (1-x)\sum_{j=0}^{N} |s_j - s^*| + (1-x)\frac{\varepsilon}{2}\sum_{j=N+1}^{\infty} x^j \\ &\leq C_1(1-x) + \frac{\varepsilon}{2} \end{aligned}$$

where $C_1 = \sum_{j=0}^{N} |s_j - s^*| \in \mathbb{R}$. It follows that for x satisfying $0 < 1 - x < \delta$, where $C_1\delta < \varepsilon/2$, we have

$$|s(x) - s^*| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

and the theorem follows.

□

A main source for power series is from Taylor's expansion. For a function f which is smooth at a point x_0, we define the *Taylor series* (or *Taylor's expansion*) of this function at x_0, $Tf(x;x_0)$, to be the power series

$$\sum_{j=0}^{\infty} \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f'(x_0)}{2!}(x-x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x-x_0)^3 + \cdots .$$

A natural question arises:

Question 1 Is $f(x) = Tf(x;x_0)$?

A general observation is that whenever $f(x)$ is given by a power series $\sum_j a_j(x-x_0)^j$, it is smooth on (x_0-R, x_0+R). Since termwise differentiation is allowed, one can show that $f^{(n)}(x_0) = n!a_n$, so the Taylor series of f at x_0 coincides with the power series, see exercise. However, in our mind the question is asked for functions given in closed form. We do know some examples that it is true, this includes $e^x, \cos x, \sin x$, etc, but that is it. After some thoughts, a cautious reader may break this question into two more precise ones:

Question 2 When f is smooth in an open interval containing x_0, does $Tf(x; x_0)$ have a positive radius of convergence?

Question 3 When $Tf(x; x_0)$ has a positive radius of convergence, is $f(x)$ equal to $Tf(x; x_0)$ in some open interval containing x_0?

It turns out the answers to both questions are negative. Without going into details, it suffices to point out that the power series of the smooth function $\exp(-1/x^2)$ at $x_0 = 0$ is identically zero. So its radius of convergence is infinity and yet this function is positive away from the origin. This example gives a negative answer to Question 3.

Let f be a function defined on (a, b) and x_0 be a point in (a, b). We call f to be *analytic* at x_0 if (i) $Tf(x; x_0)$ has a positive radius convergence and (ii) $Tf(x; x_0) = f(x)$ in an open subinterval of (a, b) which contains x_0. The function f is *analytic* on (a, b) if it is analytic at every point of (a, b). The collection of all analytic functions on (a, b), just like the collection of all continuous functions or differentiable functions, forms a vector space which is closed under product and division (provided the denominator is nonzero). In fact, it is a proper subspace of the space of all smooth functions. Unlike a general smooth function, Questions 2 and 3 always have affirmative answers for an analytic function. In this sense analytic functions are genuinely "polynomials of infinite degree". Indeed, analytic functions are precisely those functions which can be extended from the real region to the complex plane as a complex differentiable function.

Any examples of analytic functions except polynomials? Sure. Here are three:

(a) $\exp x = \sum_{j=0}^{\infty} \dfrac{x^j}{j!}, \quad x \in \mathbb{R}.$

(b) $\cos x = \sum_{j=0}^{\infty} \dfrac{(-1)^j x^{2j}}{(2j)!}, \quad x \in \mathbb{R}.$

(c) $\sin x = \sum_{j=0}^{\infty} \dfrac{(-1)^j x^{2j+1}}{(2j+1)!}, \quad x \in \mathbb{R}.$

(d) $\dfrac{1}{1+x} = \sum_{j=0}^{\infty}(-1)^j x^j, \quad x \in (-1,1).$

These three functions are analytic at $x_0 = 0$. The validity of (a), (b) and (c) was established in the course of defining the exponential and trigonometric functions, see Notes 3. For (d) it follows from the elementary formula

$$1 - x + x^2 - x^3 + \cdots + (-1)^n x^n = \frac{1+(-1)^n x^{n+1}}{1+x}$$

by letting $n \to \infty$.

It is interesting to note that some nontrivial identities can be obtained by applying Theorem 4.2 and Abel's Limit Theorem to (c). First of all, for any $x \in (0,1)$, by integrating both sides of (c) from 0 to x, we get

$$\log(1+x) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}x^j}{j}, \quad x \in (-1,1).$$

At $x = 1$, the series $\sum_{j=1}^{\infty} \dfrac{(-1)^{j+1}}{j}$ is convergent, it follows from the continuity of the logarithmic function at 1 and Abel's Limit Theorem that

$$\lim_{x\to 1^-} \log(1+x) = \lim_{x\to 1^-} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}x^j}{j} = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j}.$$

In other words, we have

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots .$$

Further, let us replace x in (c) by x^2 ($x^2 < 1$ whenever $|x| < 1$). Then we have $\dfrac{1}{1+x^2} = \sum_{j=0}^{\infty}(-1)^j x^{2j}, \quad x \in (-1,1)$. After one integration we get

$$\text{Arctan } x = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j+1}}{2j+1}.$$

Letting $x \to 1^-$ and by Abel's Limit Theorem again, we obtain the following formula which was discovered by Leibniz:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

An alternative, elementary proof of these identities without using Abel's Limit Theorem is also available, see exercise.

4.2 Newton's Binomial Theorem

This section is for optional reading.

In addition to the examples (a)-(c) in the last section, one would like to have more examples of analytic functions. We shall show that the power functions $(1+x)^\alpha$ are analytic at the origin. In order to have a detailed discussion we separate the study into two parts. In the first part we study the convergence properties of the Taylor's series of these functions. Analyticity will be established in the second part.

For any real number α, consider the power series

$$\sum_{j=0}^{\infty} c_j x^j ,$$

where

$$c_j = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-j+1)}{j!}, \quad j \in \mathbb{N},$$

and $c_0 = 1$. We call this power series a *binomial series* and c_n the *n-th binomial coefficient* of the binomial series. When $\alpha \in \{0,1,2,3,\cdots\}$, this power series becomes a polynomial. In the following we consider $\alpha \in \mathbb{R} \setminus \{0,1,2,\cdots\}$ so that it has infinitely many non-zero binomial coefficients.

Theorem 4.4. *For $\alpha \in \mathbb{R}\setminus\{0,1,2,\cdots\}$, the radius of convergence of any binomial series is 1. Moreover, it*

(i) converges absolutely at $x = \pm 1$ when $\alpha > 0$,

(ii) diverges at $x = \pm 1$ when $\alpha \le -1$,

(iii) converges conditionally at $x = 1$ and diverges at $x = -1$ when $\alpha \in (-1, 0)$.

Proof. We have

$$\lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{n+1}{\alpha - n} \right| = 1.$$

It follows that the radius of convergence is equal to 1, see no.5 in Exercises 9.4 of Text.

When $\alpha > 0$ at $x = \pm 1$, we have, for $n > \alpha + 1$

$$\left| \frac{c_{n+1}x^{n+1}}{c_n x^n} \right| = \frac{n-\alpha}{n+1} = 1 - \frac{\alpha+1}{n+1}.$$

As $\alpha > 0$, we can find some $\beta \in (1, \alpha+1)$ and n_0 such that at $x = \pm 1$

$$\left|\frac{c_{n+1}x^{n+1}}{c_n x^n}\right| < 1 - \frac{\beta}{n}, \quad \forall n \geq n_0.$$

By Raabe's Test, the binomial series converges absolutely at $x = \pm 1$.

Next, when $\alpha \leq -1$ and $x = \pm 1$,

$$\begin{aligned} |c_n x^n| = |c_n| &= \frac{|\alpha|}{1}\frac{(|\alpha|+1)}{2}\frac{(|\alpha|+2)}{3}\cdots\frac{(|\alpha|+n-1)}{n} \\ &\geq |\alpha| > 1, \end{aligned}$$

so the binomial series is divergent.

Finally, consider $\alpha \in (-1, 0)$ and $x = -1$. In this case every term of this series is positive, and we have

$$\begin{aligned} c_n x^n &= |c_n| \\ &= |\alpha|\frac{(|\alpha|+1)}{1}\frac{(|\alpha|+2)}{2}\cdots\frac{(|\alpha|+n-1)}{n-1}\frac{1}{n} \\ &\geq |\alpha|\frac{1}{n}. \end{aligned}$$

As $\sum 1/n = \infty$, the series is also divergent.

When $x = 1$, the series is alternating. We shall apply the Leibniz' Alternating Series Test. For this purpose we need to verify (i) $|c_n|$ is decreasing and (ii) $\lim_{n\to\infty} |c_n| = 0$. (i) is easy:

$$\frac{|c_{n+1}|}{|c_n|} = \frac{|\alpha|+n}{n+1} < 1,$$

as $|\alpha| < 1$. As for (ii), we observe

$$\begin{aligned} |c_n| &= \left(1 - \frac{1+\alpha}{1}\right)\left(1 - \frac{1+\alpha}{2}\right)\cdots\left(1 - \frac{1+\alpha}{n}\right) \\ &= \prod_{k=1}^{n}\left(1 - \frac{1+\alpha}{k}\right). \end{aligned}$$

To show (ii) it suffices to show

$$\log|c_n| = \sum_{j=1}^{n}\log\left(1 - \frac{1+\alpha}{j}\right) \to -\infty, \quad \text{as } n \to \infty$$

This could be proved by the Integral Test. Here we use an elementary argument.

First of all, observe that we have

$$\log(1-x) = -\sum_{j=1}^{\infty} \frac{x^j}{j}, \quad x \in (-1,1).$$

Thus,

$$\begin{aligned}
|\log(1-x)+x| &= \Big|\sum_{j=2}^{\infty} \frac{x^j}{j}\Big| \\
&\le x^2\big(\frac{1}{2}+\frac{x}{3}+\frac{x^2}{4}+\cdots\big) \\
&\le x^2(1+x+x^2+\cdots) \\
&\le \frac{x^2}{1-x}.
\end{aligned}$$

It follows that

$$\Big|\log\big(1-\frac{1+\alpha}{j}\big)+\frac{1+\alpha}{j}\Big| \le \frac{2(1+\alpha)^2}{j^2}.$$

for $j \ge 2(1+\alpha)$, that's, $1-(1+\alpha)/j \ge \frac{1}{2}$. As $\sum j^{-2}$ is convergent, this inequality shows that $-\sum \log\big(1-(1+\alpha)/j\big)$ and $\sum(1+\alpha)/j$ has the same convergence property, so $\sum \log\big(1-(1+\alpha)/j\big) = -\infty$.

$\square$

Now we come to the main result of this section. Recall when $\alpha \in \{0,1,2,3,\cdots\}$, the binomial theorem

$$(1+x)^\alpha = \sum_{j=0}^{\alpha} c_j x^j,$$

where the binomial coefficients c_j's is defined in the previous section, has been known for a long long time. Notice that for any natural number $\alpha = n$, $c_j = \binom{n}{j}$, where $j \in \{0,1,\ldots,n\}$. It was the insight of Newton who found the extension for other values of α.

Theorem 4.5. ***(Newton's Binomial Theorem)*** *For $\alpha \in \mathbb{R} \setminus \{0,1,2,3,\cdots\}$, we have*

$$(1+x)^\alpha = \sum_{j=0}^{\infty} c_j x^j, \quad \forall x \in (-1,1), \tag{4.1}$$

where the convergence is uniform on any $[a,b]$ in $(-1,1)$. Moreover, the convergence is uniform on $[-1,1]$ when $\alpha > 0$, and on $[-a,1]$, $a \in (0,1)$, when $\alpha \in (-1,0)$.

The n-th Taylor's polynomial of the function $(1+x)^\alpha$ at the origin is precisely the n-th partial sum of the binomial series for α. Thus Taylor's theorem provides a link between this function and the corresponding binomial series. In the following proof we see how the Taylor's Theorem with Integral Remainder works better than the Taylor's Theorem with Mean Value.

Proof. We will consider positive values of x first. By Taylor's Theorem with Integral Remainder in Notes 2 or (7.3.18) in Text, for any $\alpha \in \mathbb{R}$,

$$(1+x)^\alpha - \sum_{k=0}^{n} c_k x^k = (n+1)c_{n+1}\int_0^x (1+t)^{\alpha-n-1}(x-t)^n dt.$$

If $x \in [0,b],\ b<1$, we have, for $n+1>\alpha$,

$$\begin{aligned}(n+1)|c_{n+1}|&\int_0^x (1+t)^{\alpha-n-1}(x-t)^n dt \\ &\le (n+1)|c_{n+1}|\int_0^x (x-t)^n dt \le |c_{n+1}|b^{n+1}.\end{aligned}$$

By Theorem 4.4, $\sum c_j b^j$ is convergent, so $|c_{n+1}|b^{n+1}$ tends to 0 as $n\to\infty$. Thus, for every $\varepsilon>0$, there exists n_0 such that for any $\alpha\in\mathbb{R}$,

$$|(1+x)^\alpha - \sum_{j=0}^{n} c_j x^j| \le |c_{n+1}|b^{n+1} < \varepsilon,\ \forall n\ge n_0,\ \forall x\in[0,b].$$

We have shown that for any real α, (4.1) holds and the convergence is uniform on $[0,b]$ for any $b<1$.

Next, consider negative values of x. When $x\in[a,0],\ a>-1$, we use the Mean-Value Theorem to get

$$\begin{aligned}\left|c_{n+1}\int_0^x (1+t)^{\alpha-n-1}(x-t)^n dt\right| &= (n+1)|c_{n+1}|\int_x^0 (1+t)^{\alpha-n-1}(|x|+t)^n dt \\ &= (n+1)|c_{n+1}|(1+\xi x)^{\alpha-n-1}(|x|-\xi|x|)^n\int_x^0 dt \\ &= (n+1)|c_{n+1}|(1+\xi x)^{\alpha-1}\Big(\frac{1-\xi}{1+\xi x}\Big)^n|x|^{n+1},\end{aligned}$$

for some $\xi\in(0,1)$. As $x\in(-1,0),\ 1-\xi\le 1+\xi x$, so

$$\left|c_{n+1}\int_0^x (1+t)^{\alpha-n-1}(x-t)^n dt\right| \le M(n+1)|c_{n+1}||x|^{n+1},$$

where $M = \sup\Big\{(1+\xi x)^{\alpha-1}\ :\ \xi\in(0,1),\ x\in[a,0]\Big\}$. By Theorem 4.4 the radius of convergence of $\sum c_j x^j$ is equal to one. It implies that the radius of convergence of the series $\sum(j+1)c_jx^j$ is also equal to one. Thus $\sum(j+1)c_ja^j$ converges absolutely and consequently $(n+1)c_{n+1}|a^{n+1}| \to 0$ as $n \to \infty$. As before we conclude that $\sum c_jx^j$ converges uniformly to $(1+x)^\alpha$ on $[a,0]$ for any real α.

When $\alpha > 0$,
$$\left|\frac{c_{n+1}}{c_n}\right| = \frac{n-\alpha}{n+1} = 1 - \frac{1+\alpha}{n+1},$$
for all $n > \alpha$. When $\alpha > 0$, we can fix some $\beta \in (0,\alpha)$ such that
$$\left|\frac{c_{n+1}}{c_n}\right| \geq 1 - \frac{1+\beta}{n}$$
for all $n \geq n_1$. By Raabe's test, $\sum_1^\infty |c_n|$ converges. It follows from the M-test that $\sum_1^\infty c_nx^n$ converges uniformly on $[-1,1]$.

Finally, when $\alpha \in (-1,0)$, the series is alternating. For $\varepsilon > 0$, there exists n_0 such that $|c_n| < \varepsilon$ for all $n \geq n_0$. Using this fact, for $x \in [0,1]$, $|\sum_{j=m+1}^n c_jx^j| < \varepsilon, \forall n,m \geq n_0, \forall x \in [0,1]$. Hence the series converges uniformly on $[0,1]$.

□

4.3 Euler's Formula for Negative Powers

This section is for optional reading.

Although we have proved many criteria on the convergence of infinite series of numbers, seldom did we evaluate their sums. In this section we discuss a well-known summation formula for negative powers discovered by Euler in 1735, when he was twenty-eight.

From high school we learnt how to sum up a geometric progression. It led to the formula
$$\frac{1}{1-a} = 1 + a + a^2 + a^3 + \cdots, \qquad a \in (-1,1).$$
In particular, taking $a = 1/2$ and $-1/2$ yields
$$2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots,$$
and
$$\frac{2}{3} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots.$$

From the previous sections we know more, for instance,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots ,$$

and

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots .$$

Euler's formula gives a closed form for the sum

$$E_k = \sum_{n=1}^{\infty} \frac{1}{n^k},$$

when k is an even number. For instance, we have

$$\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots .$$

Euler discovered this formula by a wonderful method based on analog thinking. To describe it we start with some simple facts on algebraic equations. Let a polynomial of degree n be given by

$$p(x) = 1 + a_1x + a_2x^2 + \cdots + a_nx^n.$$

Recall that x_0 is a root of p if $p(x_0) = 0$. The multiplicity of a root is defined to be the number m that satisfies $p(x_0) = p'(x_0) = \cdots = p^{(m-1)}(x_0) = 0$ but $p^{(m)}(x_0) \neq 0$. A root of multiplicity one is called a simple root and a root of multiplicity two is called a double root. The multiplicity appears in the power in the factorization of the polynomial. For instance, the polynomial $x^3 + x^2 - x - 1$ has a double root -1 and a simple root 1. We have the factorization

$$x^3 + x^2 - x - 1 = (x + 1)^2(x - 1).$$

Of course, a polynomial may admit complex roots so complete factorization over the real field is not always possible. For instance, for the polynomial x^3-6x^2+x-6 we stop at $(x^2 + 1)(x - 6)$. However, suppose now that the polynomial of degree n $p(x)$ has exactly n many real simple roots $\alpha_1, \cdots, \alpha_n \neq 0$. By comparing the coefficients of the constant term we have the factorization formula

$$1 + a_1x + a_2x^2 + \cdots + a_nx^n = \prod_{k=1}^{n} \Big(1 - \frac{x}{\alpha_k}\Big).$$

By comparing the coefficients of $x^k, k = 1, \cdots, n$ from both sides, we get

$$\begin{aligned} a_1 &= -\sum_j \frac{1}{\alpha_j}, \\ a_2 &= \sum_{i<j} \frac{1}{\alpha_i \alpha_j}, \end{aligned}$$

and

$$a_k = (-1)^k \sum_{i_1 < \cdots < i_k} \frac{1}{\alpha_{i_1} \cdots \alpha_{i_k}}, \quad k = 1, \cdots, n,$$

in general.

Now, consider the Taylor expansion for the sine function,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots,$$

which is valid for all x in $\mathbb{R}$. The function

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

(set $\sin x/x = 1$ at $x = 0$) is smooth on $\mathbb{R}$. Euler boldly regarded $\sin x/x$ as a polynomial of infinite degree and asserted that all roots of $\sin x/x = 0$ are real, simple and given by $\pm k\pi, k \geq 1$. Moreover, parallel to the factorization above, one has

$$\frac{\sin x}{x} = \prod_{k=1}^{\infty} \left(1 - \frac{x}{k\pi}\right)\left(1 + \frac{x}{k\pi}\right) = \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2\pi^2}\right) \tag{4.2}$$

By comparing the coefficients of x^2 in this infinite product with the Taylor's series of $\sin x/x$, he got

$$\frac{\pi^2}{6} = \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

By comparing the coefficients of x^4, he got

$$\frac{\pi^4}{90} = \sum_{k=1}^{\infty} \frac{1}{k^4}$$

after some manipulations. Going up step by step, all E_{2k} could be computed by looking at the coefficients of x^{2m} together with $E_{2(k-1)}, \cdots$, and E_2.

The formula (4.2) was first obtained in a formal way. Years later, Euler justified it in rigorous terms. His proof used complex variables but the essential idea could be carried entirely out in the real field. There are other proofs using, for instance, Fourier series. The following "real" proof is taken from O. Hijab, Introduction to Calculus and Classical Analysis, Springer-Verlag, 2007. In this formula the sine function is replaced by the hyperbolic sine to the same effect.

Proposition 4.6.

$$\frac{\sinh \pi x}{\pi x} = \prod_{n=1}^{\infty}\left(1+\frac{x^2}{n^2}\right), \quad \forall x \in \mathbb{R}. \tag{4.3}$$

(Set $\sinh \pi x/\pi x = 1$ at $x = 0$.) Recall that $\sinh x = (e^x - e^{-x})/2$ is the hyperbolic sine function.

Proof. We start with an identity of factorization: For $a, b > 0$,

$$a^{2n} - b^{2n} = (a^2 - b^2)\prod_{k=1}^{n-1}\left(a^2 - 2ab\cos\frac{k\pi}{n} + b^2\right)$$

(Exercise). So

$$\begin{aligned}
\frac{\left(1+\frac{\pi x}{2n}\right)^{2n} - \left(1-\frac{\pi x}{2n}\right)^{2n}}{2\pi x} &= \frac{\left(1+\frac{\pi x}{2n}\right)^{2} - \left(1-\frac{\pi x}{2n}\right)^{2}}{2\pi x} \times \\
&\prod_{k=1}^{n-1}\left[\left(1+\frac{\pi x}{2n}\right)^2 - 2\left(1+\frac{\pi x}{2n}\right)\left(1-\frac{\pi x}{2n}\right)\cos\frac{k\pi}{n} + \left(1-\frac{\pi x}{2n}\right)^2\right] \\
&= \frac{1}{n}\prod_{k=1}^{n-1}\left[2\left(1+\frac{\pi^2x^2}{4n^2}\right) - 2\left(1-\frac{\pi^2x^2}{4n^2}\right)\cos\frac{k\pi}{n}\right] \\
&= \frac{1}{n}\prod_{k=1}^{n-1}\left[2\left(1+\frac{\pi^2x^2}{4n^2}\right)\left(\sin^2 x + \cos^2 x\right) - 2\left(1-\frac{\pi^2x^2}{4n^2}\right)\left(\cos^2\frac{k\pi}{2n} - \sin^2\frac{k\pi}{2n}\right)\right] \\
&= \frac{1}{n}\prod_{k=1}^{n-1}\left(4\sin^2\frac{k\pi}{2n} + \frac{\pi^2x^2}{n^2}\cos^2\frac{k\pi}{2n}\right)
\end{aligned}$$

Letting $x \to 0$, an application of L'Hospital's rule yields

$$1 = \frac{1}{n}\prod_{k=1}^{n-1} 4\sin^2\frac{k\pi}{2n}.$$

By termwise division,

$$\frac{\left(1+\frac{\pi x}{2n}\right)^{2n} - \left(1-\frac{\pi x}{2n}\right)^{2n}}{2\pi x} = \prod_{k=1}^{n-1}\left[1 + \frac{x^2}{k^2}\varphi\left(\frac{k\pi}{2n}\right)\right]$$

where $\varphi(t) = t^2 \cot^2 t$. Using $\tan t \geq t$, we see that $\varphi(t) \leq 1$ on $(0, \pi/2)$. Therefore,

$$\begin{aligned}
\frac{\left(1+\frac{\pi x}{2n}\right)^{2n} - \left(1-\frac{\pi x}{2n}\right)^{2n}}{2\pi x} &\leq \prod_{k=1}^{n-1}\left(1+\frac{x^2}{k^2}\right) \\
&\leq \prod_{k=1}^{\infty}\left(1+\frac{x^2}{k^2}\right).
\end{aligned}$$

Let $n \to \infty$,

$$\frac{\sinh \pi x}{\pi x} \leq \prod_{k=1}^{\infty}\left(1+\frac{x^2}{k^2}\right).$$

On the other hand, fix n_1 so that

$$\frac{\left(1+\frac{\pi x}{2n}\right)^{2n} - \left(1-\frac{\pi x}{2n}\right)^{2n}}{2\pi x} \geq \prod_{k=1}^{n_1-1}\left(1+\frac{x^2}{k^2}\varphi\left(\frac{k\pi}{2n}\right)\right), \quad \forall n \geq n_1.$$

Letting $n \to \infty$,

$$\frac{\sinh \pi x}{\pi x} \geq \prod_{k=1}^{n_1-1}\left(1+\frac{x^2}{k^2}\right)$$

and

$$\frac{\sinh \pi x}{\pi x} \geq \prod_{k=1}^{\infty}\left(1+\frac{x^2}{k^2}\right)$$

follows by letting $n_1 \to \infty$.

$\square$

Taking log of both sides of (4.3) and using the continuity of the logarithmic function, we have

$$\begin{aligned}
\log \sinh \pi x - \log(\pi x) &= \log \lim_{n\to\infty}\prod_{k=1}^{n}\left(1+\frac{x^2}{k^2}\right) \\
&= \lim_{n\to\infty} \log \prod_{k=1}^{n}\left(1+\frac{x^2}{k^2}\right) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \log\left(1+\frac{x^2}{k^2}\right) \\
&= \sum_{k=1}^{\infty} \log\left(1+\frac{x^2}{k^2}\right).
\end{aligned}$$

We claim the series on the right hand side is uniformly convergent on $(0, M]$ for all $M > 0$. Indeed, by the mean-value theorem

$$\log\left(1+\frac{x^2}{k^2}\right) = \frac{k^2}{k^2+c^2}\frac{x^2}{k^2},$$

for some c between 1 and x^2/k^2. Therefore,

$$0 < \log\left(1+\frac{x^2}{k^2}\right) \le \frac{M^2}{k^2}, \quad \forall x \in (0, M].$$

Taking $b_k = M^2/k^2$ and applying M-Test we obtain the result as claimed. Furthermore, the series obtained by differentiating $\sum_k \log(1 + x^2/k^2)$ whose general term is

$$\frac{k^2}{k^2+x^2}\frac{2x}{k^2}$$

also converges uniformly on every $(0, M]$. By the "exchange theorem" it is legal to differentiate both sides of

$$\log\sinh\pi x - \log\pi x = \sum_{k=1}^{\infty}\log\left(1+\frac{x^2}{k^2}\right)$$

to get

$$\pi\coth\pi x - \frac{1}{x} = \sum_{n=1}^{\infty}\frac{2x}{n^2+x^2}, \quad x \neq 0,$$

or

$$\frac{\pi\coth\pi\sqrt{x}}{\sqrt{x}} - \frac{1}{x} = \sum_{n=1}^{\infty}\frac{2}{n^2+x}, \quad x > 0,$$

We would like to expand the left hand side of this expression into a Taylor series. First, consider the function

$$\tau(x) = \begin{cases} \dfrac{x}{1-e^{-x}}, & x \neq 0 \\ 1, & x = 0 \end{cases}.$$

This function is the reciprocal of the power series

$$\frac{1-e^{-x}}{x} = \sum_{k=0}^{\infty}(-1)^k\frac{x^k}{(k+1)!}, \quad \forall x \in \mathbb{R}.$$

According to a problem in Ex 12, we can expand it as a power series at 0,

$$\tau(x) = \sum_{n=0}^{\infty}\frac{\beta_n}{n!}\,x^n,$$

where $\beta_0 = 1,\ \beta_1 = 1/2,\ \beta_2 = 1/6,\ \beta_3 = 0,\ \beta_4 = -1/30, \cdots$, etc. Observing that

$$\frac{x}{2} \coth \frac{x}{2} = \tau(x) - \frac{x}{2},$$

we have

$$\frac{x}{2} \coth \left(\frac{x}{2}\right) = 1 + \sum_{n=2}^{\infty} \frac{\beta_n}{n!}\ x^n.$$

As the left hand side of this identity is an even function, $\beta_{2n+1} = 0, \forall n \geq 1$ and

$$\frac{x}{2} \coth \left(\frac{x}{2}\right) - 1 = \sum_{n=1}^{\infty} \frac{\beta_{2n}}{(2n)!}\ x^{2n}.$$

Finally we conclude

$$\frac{1}{2} \sum_{n=1}^{\infty} \frac{\beta_{2n}}{(2n)!} (2\pi)^{2n} x^{n-1} = \sum_{n=1}^{\infty} \frac{1}{n^2 + x}.$$

By differentiating both sides of this identity $k - 1$ many times and then setting $x = 0$ (uniform convergence after differentiation is easy to verify), we finally obtain

Theorem 4.7 (Euler's Formula). *For all $k \geq 1$,*

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1}}{2} \frac{\beta_{2k}}{(2k)!} (2\pi)^{2k}.$$

In the exercise you are asked to show that β_k is equal to b_k, the k-th Bernoulli number introduced in Notes 2 for $k \geq 2$. Despite the effort of many mathematicians, little is known for E_k when k is odd. It was proved as late as 1979 that E_3 is an irrational number. You may look up the expository paper, "Euler and his work on infinite series", Bulletin of AMS, 515-539, 2007, by V.S. Varadarajan for more.

It is interesting to observe that E_k are special values of the Riemann zeta function ζ defined by

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}, \quad x > 1.$$

It is not hard to see that this series converges uniformly on $[a, \infty)$ for each $a > 1$ and ζ is smooth on $(1, \infty)$. (In fact, the zeta function can be defined in $z \in \mathbb{C}/\{1\}$.) Thus we have $E_k = \zeta(k)$. It is related to the Gamma function Γ by the relation

$$\zeta(x) = \frac{1}{\Gamma(x)} \int_0^{\infty} \frac{t^{x-1}}{e^t - 1} dt.$$

Using this relation one could obtain another proof of Euler's formula. Finally, we point out that the the zeta function has deep relationship with prime numbers. The following identity was found by Euler:

$$\zeta(x) = \frac{1}{\prod_p \left(1 - \frac{1}{p^x}\right)},$$

where the product is taken over all prime numbers. As $\lim_{x \to 1^+} \zeta(x) = \infty$, this identity shows that there are infinitely many prime numbers. This result essentially opens up a new branch of mathematics called analytic number theory.